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Measures and measurement from data, advanced mathematical basements.

English version

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Contents

Summary:
This communication concerns numbers and how to use them in scientific processes and terms. Naturally the number has a history of its own, as do the disciplines that use it with success and to good purpose, while there are others, drawn by such success, that are affected by number appeal and demonstrate a recurrent form of mimetism.
However, the number is too rich and precious a tool for its rules and properties to be ignored. This exposé deals with the "theory of measurement" and attempts to clarify the rules of numerical assignment to observations and other entities, including many concepts that belong to the sphere of the human sciences. The method of successive intervals is one application of measurement to the case of responses collected according to ordered categories of scales. It is then presented by way of two applications in scientific research related to transport, i.e. comfort in ergonomics and nuisance due to noise.

English version in the published Congress Proceedings.

key-words : *number, numerical representation, scaling, numerical assignment, measurement, psychophysics, law of categorical judgments, successive intervals, noise annoyance, comfort evaluation*

I - Introduction.

Current observations show that the number and numerical methods of expression are used in the terms and results of several scientific disciplines, especially for statistical treatments and when seeking "laws". Consequently, the use of the number has sometimes become standard practice and conformity with the rules of calculation is not always respected. This presentation seeks to stir awareness of the subject and of the precautions that must be taken when choosing to express oneself by using the panoply of numerical tools. The presentation will be concluded with examples of comfort and ergonomics of car interior (Wang & Maurin), and transport generated noise annoyance in the framework of indicators of impacts caused by nuisances (Maurin 2003 a).

II - Science and the number.

It seems appropriate to start with a few historical reminders, including one from Galileo (Il Saggiatore), who stated that "*The language of nature is mathematics*" at a time, it should be emphasised, when mathematics was in its infancy. It has also become traditional practice to state that many others have used this aphorism since (Lévy-Leblond).

II.1 - A little history and the place of the number.

Going back even further in time, the Bible not only includes the Gospel of Saint John, which starts with the well-known "*In the beginning was the word*", there is also the **"Book of Numbers"** of the **Penta**teuch at the beginning of the Old Testament, and even before the **first** night after the creation of the sky and earth, then the **second**, etc. until the **seventh** day, the day of rest.
This clearly shows that the number with its associated mathematical universe is a component that man has used since the beginning of his history. On reading Herodotus (or Proclus, (Guilbaud)) we return to the creation (or discovery) of *geo-metry* whose role was to stake out land every time the floods of the Nile subsided. Both the Rhind Papyrus and the Tablets of Hammourabi date from about 1800 BC and Thales (500 BC) was considered as the first to apply demonstrative reasoning in the world of mathematical entities.
In addition to this early history (which is not exhaustive regarding other civilisations, such as China, India, Central America, etc.) we should mention the success of the number for the needs of Physics, especially with Galileo (1620), the founder of classical Physics. Although it is more recent, this use of the number has nonetheless considerably marked the last four centuries, and it has contributed towards recognising the fact that, for both matter (physics) and space (geometry), the numerical representation of observations and the mathematical reasoning that accompanies it are "natural", probably indispensable and extremely productive.

We can also add that the mathematical development of physics has not gone unnoticed, and it is possible to write a parallel history for the mimetism of the human sciences regarding the recurrent use of the number. Leonardo de Vinci used the language of numbers for intellectual operations a century before Galileo (Perse), while in the 18th century Cramer and Daniel Bernoulli formulated a model of the utility of money (Roberts). Wolff, one of Kant's teachers, then wished to use the advantages of the number (Gusdorf), while Bonnet, the inventor of the short-lived "psychometre" built on the model of the "geometer" (Perse) after which, in the 19th century, Jeremy Bentham developed his *felicity calculus*. Then came Auguste Comte, who created sociology, giving it its initial name of social physics; Fechner, who founded psychophysics, giving it an elegant ecumenical name, while Galton achieved as much for psychometry with terms very similar to those used by Wolff and Vinci, while also giving us the noun "psychometrician". We can also go far back into antiquity and mention the myth of Tiresias the Seer of Thebes (present as early as the Odyssey) who did not hesitate to use the number to compare the intensities of two sensations, much to the displeasure of Hera.

II.2 – The use of the number and the "*number appeal*".

The reader perhaps knows other examples of this recurrent use of the number. It has also been said that "*the number permits discourse that goes beyond oral and qualitative expression*" (Boudon). This mimetism of the numeric in these predominantly human disciplines could also be called "*number appeal*" to mark this regularly resurgent attraction and which leads directly to questions on the use of the number and the correction of the rules applied to it for such uses.

An initial example of the assignment of numbers to things can be given by the use of numbers for the simple purpose of identifying the players of a football team and French "*départments*" (many others exist). In the former case, nobody would think of saying that a player with number 10 is five times better than that with jersey number 2; or that the department of the Gironde (33) is worth three times the Aude (11), or that the Pyrénées Atlantiques (64) is worth four times the Charente (16). Obviously, this type of use appears unthinkable, though since numbering is so widespread, the abusive use of the number and its poorly understood rules are often well hidden, laying traps for the unsuspecting.

This also holds for the discipline of physics itself. In acoustics, for example, noise levels are expressed by numbers (in decibels), but one should not deduce from this that if two independent noise levels of 70 each are superposed, the result would be a level of 140. Indeed, if noise levels are quantified correctly, their definition does not correspond to a rule of addition vis-à-vis the numbers assigned to them (Maurin 1999, 2003 b). Furthermore, although every acoustician is aware of the fact, the statistical calculations of noise in the environment that are performed are often done so using software based on the addition of the numbers processed, and they are rarely programmed for other purposes. Likewise for calculations of pH in chemistry which have a formal mathematical status very close to that of levels (logarithms are used in both cases).

III – Models and scaling.

Strictly speaking, mathematics and physics are outside our scope, though it is important to examine their interactions when developing models.

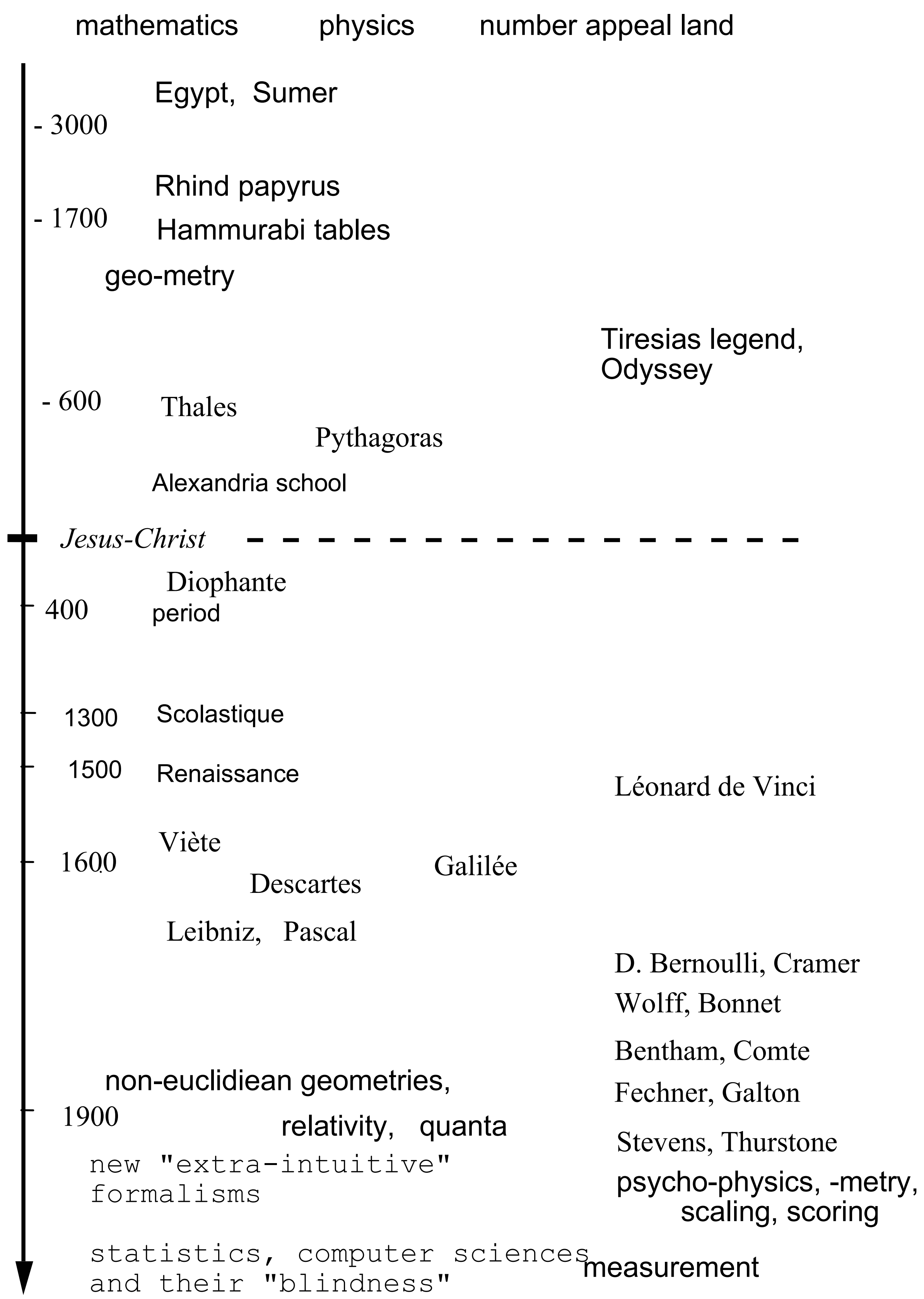

Figure 1 - The number history (Abstract)

III.1 – The zetetic approach to models.

A model consists in producing a mathematical and numerical representation of the physical world (or more modestly a part of the world!), and to achieve this it gathers within the same set of relations the magnitudes that interact in a phenomenon observed. This developed after what could be termed the revolution of zetetic analysis introduced by Viète (1591), (Legrand). This entails representing known magnitudes by a literal symbol whose role is to represent a numerical value and also to include within the same relation symbols of magnitudes whose values are unknown by considering the problem as being resolved, and by applying to them the same calculation rules as for the symbols of known quantities, without distinction (the use of "indeterminates" dates back to Diophantes (Guilbaud), although they were integers or rationals). Technically, this then resulted in algebraic systems whose solutions are the numerical values of unknown magnitudes.

Literal symbolism is an essential convenience of algebraic symbolism that was taken up and perfected by Descartes to develop analytical geometry (1637), or the combination of geometry and algebra, which had been dealt with separately up to then. Physical modelling made use of this later on with the symbolisation of known and unknown values in the same equations. Shortly afterwards this approach and Descartes' finite and deterministic vision were completed by the calculation of probabilities of Pascal and Fermat (1655), and the infinitesimal calculus of Leibniz and Newton (1680). Whatever the case, a step of numerical resolution must be added, using more or less sophisticated techniques that are now increasingly computerised.

III.2 – The response of man and his numbering

This approach was then taken up by the human sciences. Looking back once again to the classical world, one often hears repeated Protagoras' aphorism stating that "*man is the measure of all things*". We know that Protagoras was a leader of the Sophists and that this form of thinking can lead to many digressions, though here we take them literally by considering man's response to the sensations and judgements to which he is subject. This is the case in particular when seeking to numerise and model man's response. Three main types of model and numbering in psychophysics exist, in addition to Fechner's preliminary foray, which remains largely tautological.

a) The most immediate is the approach formulated by Stevens since it explicitly requires a numerical response to subjects by conforming to the rule of numerical proportions. This is the *magnitude estimation* method, and we know the success of this approach at the origin of Stevens' Power Laws for sensation, (in this respect Tiresias was a precursor of *magnitude estimation*).

a2) Stevens has less procedural successors (Gescheider), for example, the simple use of a line in a questionnaire demanding that the subject place a "tick" on it as a measure of the magnitude of the sensation they feel. Other protocols exist in which numbers can be noted directly though generally these remain rudimentary regarding the status and use of the numbers involved (Annett, Wang & Monnier).

b) Another very widespread mode of response consists in asking subjects to answer by giving

preferences or by making binary choices between two simulations (in the broad meaning). This is the pairs comparison approach that can be modelled in order to calculate a numerical value for each stimulation on a common continuum, and, for example, there is Thurstone's law of comparative judgments (Torgerson).
c) When individual stimulations and their comparison become too numerous, a simplification is made by setting a scale of ordered categories of responses, the bounds being considered as dummy stimulations, and the stimulations are only compared in relation to these bounds. This is the idea behind Thurstone's (and then Saffir's) adaptation for developing the law of categorical judgments (Torgerson).

These models and their associated numerical resolutions are a recurrent way of subscribing to number appeal and it can be added that in this case it is used as a springboard for the numerical relationship appeal in the well-known nomothetic approach of science. The idea and quest for a numerical correspondence between physical excitation and numerised response is one of the initial intentions of Fechner's psychophysics while "*nomothetic*" features in Stevens (1971).

III.3 - Scaling.
Scaling consists in using this literal and algebraic approach to what appears to be the sole case of magnitudes in the human sciences, with different numerical and statistical calculations thrown in. It is perhaps a more accurate description of applications with a large number of variables as a function of adjustments, such as multidimensionnal scaling (Kruskal, Sheppard, Torgerson), whereas *scoring* better describes the modelling of a lower number of variables (Agresti).

This is perhaps why the poet and thinker Paul Valéry said "*Descartes is surely one of the men most responsible for the allure and physiognomy of the modern era, which is characterised in particular by what I would call the quantification of life*" (Variété, quoted by Diéguez). After such a declaration, Descartes finds himself involved in the vertigo of *number appeal* in the life sciences, whereas he used analysis only in the sciences of matter and space. However, it is true that Viète and Descartes opened the way to the formulation of mathematical models including unknowns in a literal form within reach of numbers.

III.4 – Numerical assignment and the questions it raises.
The consequence of the previous reminders is that the number is firmly fixed in the history of humanity and its vision of the world, and it has developed myriad techniques in many fields to assign numerical values to objects, entities and other observations. The zetetic and algebraic process was justified when applied to geometric and physical magnitudes whose existence was almost certain, but its success and the habitual use of this practice has perhaps blunted caution as to the pertinence of its use in other disciplines, making way for a certain amount of mimetism. Moreover, one may wonder in passing whether calculations made using numbers and symbols to represent all kinds of magnitudes do not result in endowing them with legitimacy. Is not the inclusion of these magnitudes a means of bending a method to one's purpose by lending it substance through numbers? What is more, isn't this a kind of inversion of the relationship between existing magnitudes and images made mathematical? And doesn't this now risk being amplified by the development of statistical processes on the

one hand, and by software on the other, of new machines for crunching numbers though which hardly give the impression of questioning their status. Some may occasionally fear that computer processes inhibit (or flatter) the new form of scientific spirit of our times.

Since the pressure related to assigning the number to different magnitudes and its use in scientific discourse is obvious, this exposé is an opportunity for recalling the necessity of conforming to meaningful syntax during the many discourses given in the framework of the numerical universe.

IV – The measurement theory.

The theory of measurement essentially dates from the second half of the 20th century. It revigorates numerical assignment and responds to the previous questions on scaling. Indeed, it first ensures that numerical representations exist instead of simply postulating them by way of literal algebraic symbols, and to do this it relies on the qualitative and consequently pre-numeric examination of data e_i of a group E of observations.

IV.1 - Representation.

a) Qualitative examination consists in identifying the possible particular relations (qualitative, ensemblist) manifested by these data, as do, for example, relations of order between observations, judgements, preferences, and rules of composition when two observations can be assembled (which can lead to a homomorphic rule with addition), etc. They are formalised by binary relations $R_{(j)}$ and the ensemble E = {E, $R_{(j)}$} constitutes an ERS, (empirical relational system).

b) It then entails finding a homomorphic structure in the range of mathematical structures, that is to say an ensemble of points or formal elements that verify relations that are homomorphic or of the "same kind". This structure is called an NRS or numerical relational system, and it entails establishing a correspondence μ between the two relational systems. This is the object of a theorem of representation, that is to say the existence of a representation, in the same way as theorems of existence exist in mathematics (solutions of an algebraic equation, a differential equation, a functional equation, and so forth). Thus correspondence exists when the relations of the ERS are effectively verified, and the verifications can be made on observing data e_i.

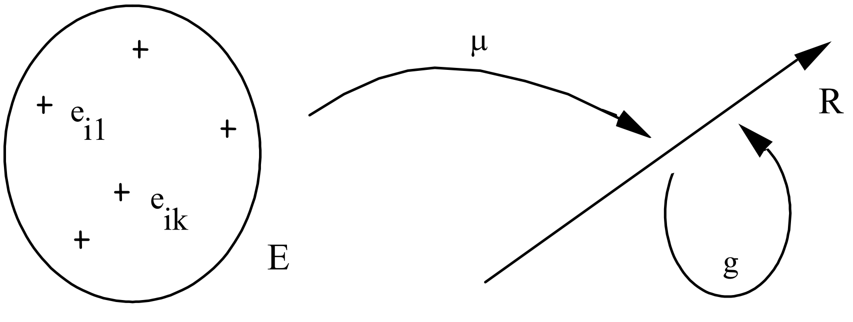

Figure 2, homomorphism between ERS and NRS of real number R

c) When the NRS is the line R of the real numbers, we are sure of the existence of a numerical representation $\mu(e_i)$ of the initial observations with what is called a scale of measurement. It should be noted that when a correspondence exists, it conforms to the all the qualitative relations observed on the data by its construction (Roberts, Suppes & Zinnes).

IV.2 – Characterisation of the scale, the uniqueness.

When demonstrating existence, we often obtain several representations that respond to the question. In the simplest cases, if μ is a correspondence that responds to the question, composition g o μ is another with a transformation g of R onto R which belongs to a sub-group G of transformations (thus considered permissible), and the scale of measurement is characterised by G. The most usual cases are listed in the following table and are types of scale known elsewhere, especially since the psychophysicist Stevens (Roberts). However, here they are introduced according to the logic of representations of measurement and their characteristics. (There is also the more complex case where the correspondences μ are listed without the presence of a sub-group and its action on correspondences μ).

Sub-group G of transformations	type of scale	
identity, neutral element of G	absolute scale	
$g(x) = c\,x$, c positive	ratio scales	(cardinal)
$g(x) = c\,x + b$, c positive	interval scale	(cardinal)
g increasing	ordinal scale	
g bijective	nominal scale	

table 1, the main types of scales characterised by a sub-group of permissible transformations.

IV.3 - The meaningfulness of numerical statements.

Naturally, we formulate numerical images $\mu(e_i)$ to express results with these numbers, and these statements are used as the basis for developing the notion of meaningfulness. A statement comprising numerical values is considered meaningful if it remains invariant vis-à-vis the action of transformation g characterising the scale (the transformations introduced for the uniqueness). For example, the relation of the order $\mu(e_i) > \mu(e_k)$ is meaningful for any ordinal, interval and ratio scale; however, statement $\mu(e_i) = 3\ \mu(e_k)$ does not remain invariant for interval and ordinal scales.

Meaningfulness extends to statistical treatments and the statements that use them. For example, the arithmetical average has a meaning for values measured on interval and ratio scales but none for ordinal scales, although the median is meaningful for the three types of ordinal, interval and ratio scale.

IV.4 - Other developments.

a) Numerical values as such are only useful if the correspondence μ on which one is working is specified in full. This can lead to questions to resolve more classical numerical systems, such as for scaling (Coombs Dawes & Tversky).

b) The presentation above is implicitly oriented towards the fact that observations e_i of E belong to a univariate ensemble, for example sensations that rely only on one magnitude of sensation. However, this theory has also moved rapidly towards the "conjoint measurement" of observations belonging to E ensembles having a bi- or multivariate structure, such as exposures to two independent stimulations with which it is possible to compare (at the level of the pre-numerical relations of ERS) combinations of multiple exposures, mainly with preferences between situations combined ad hoc. Once again, sets of measurement conditions permit the formulation of measurement scales (existence and uniqueness) for each of the univariate dimensions that subtend the multivariate structure of the data (Pfanzagl, Roberts). This is the extension that has given rise to considerable refinements since the additive conjoint measurement of Luce and Tuckey, (Bouyssou & Pirlot).

c) The presentation of measurement became more refined in the sixties (Roberts, Suppes & Zinnes). The essence of this approach lies in the effort made to ensure the existence of a numerical representation, in view of qualitative, ensemblist and relational conditions of data. These conditions are also called measurement axioms. It should be noted that the theorems remain in the realm of theory while the verification of axioms is done very practically on the level of data and observations.

Regarding this it rapidly became clear that data were contaminated by fluctuations (like all observations in practice). The result of this is that conformity with these conditions to establish a theorem of existence is too rigid, since the conditions cannot take these fluctuations into account. Another result is that scales can exist whatever the case, although they are hidden by noise.

It was to solve this difficulty that shortly afterwards the notion of Probablistic Measurement (Falmagne 1976) was introduced. In substance this is the probabilistic transcription of axioms with statistical reasoning in order to admit the hypothesis that a scale of measurement exists in spite of non-conformity, in the strict meaning of the term, with axiomatic conditions. It was seen that probabilistic measurement could be applied well to the type of data collected in the Human Sciences, with responses in many areas (Luce & Suppes) that are often data in binary or categorical form, § III.2, (Falmagne 1978 a and b, Hamerle & Tutz, Mausfeld).

Thus it can be seen that the theory of measurement is above all devoted to the existence of a numerical representation that conforms to the qualitative properties explicated by the observation of data. Naturally, the sciences of matter and space, which have circumscribed and targeted their magnitudes with scientific consensus, are not concerned; however this exigency of the existence of representation allows other disciplines to join the march towards the number (evidently, if necessary), without succumbing to the excesses of *number appeal*.

V – Successive intervals.

The theory of measurement has good applications in psychophysics (Falmagne 1985, Roberts). We shall continue in the framework of human response according to a scale of ordered categories, § III.2.c, and present an appropriate method called the successive interval method.

V.1 – Categorical responses.

The people questioned are subjected to different stimulations noted s_i for an index i from 1 to I, and they must respond as a function of a scale of ordered categories C_j for j from 1 to J; number J is fixed by the operator and is generally from 4 to 11, without forgetting the "*magical 7*", (Miller). Under these conditions it is possible to collect data in the form of a contingency table with the categories of response at the heads of the columns, the stimuli at the beginning of the line, and the different numbers n_{ij} of persons that respond C_j when subjected to exposure s_i, (table 2).

It is very usual to use a single numerical value c_j for each category C_j in multidimensional scaling applications, or in a more artificial way to take c_j as being equal to rank j of the category in the scale. The comment *"to the theorist, however, the whole business is a bit hair-raising, ..., because there is nothing about the procedure to prevent one from labelling the categories by any other increasing sequence of numbers"* (Luce & Galanter) has not sufficed to stem the success of these methods.

	C_1	C_2		C_j		C_J	
s_1							
s_i				n_{ij}			n_{i+}
s_I							
				n_{+j}			

Table 2, the contingency table of responses in $\{C_j, s_i\}$

V.2 – A mode of representation by intervals.

An essential change of view consists in representing each category by an interval between two bounds $[t_{j-1}, t_j]$. This representation naturally conforms more to the nature of the categories and is that used by Thurstone for categorical judgements.

Furthermore, in a general way Thurstonian approaches (§ III.2.b et c) consider the response to a stimulation s_i as a random variable $\mathbf{S}_i$. In the paired comparisons the preference of s_i to s_k is modelled by the probability of the event $\{\mathbf{S}_i - \mathbf{S}_k \geq 0\}$. In the categorical judgements random variables $\mathbf{T}_j$ are introduced for the bounds, and response C_j when exposed to s_i is modelled by the probability of the event $\{\mathbf{T}_{j-1} \leq \mathbf{S}_i \leq \mathbf{T}_j\}$, (Maurin 2003 b, Torgerson). The psychophysical model uses normal laws for the random variables $\mathbf{S}_i$ and $\mathbf{T}_j$ (normal laws constitute one of the very rare technical occasions where one knows how to calculate the probability of events $\{\mathbf{S}_i \leq \mathbf{T}_j\}$).

V.3 – The measurement of scales of categories.

The application of measurement of this type of response is due to Adams and Messick in 1958, and is known as successive intervals (Adams & Messick, Maurin 1986 b, Suppes & Zinnes). It is presented in the form of several steps.

a) Firstly, Adams and Messick took another technical option for the laws of random variables. On the one hand the centred variables $(\mathbf{S}_i - \mu_i)/\sigma_i$ reduced by $\mu_i = E(\mathbf{S}_i)$ and $\sigma_i^2 = var(\mathbf{S}_i)$ obey a common law as a function of the normal F distribution or not, and on the other hand $\mathbf{T}_j$ obeys Dirac's discrete distributions in t_j. This is another option with which it is also possible to explain probability p_{ij} of $\{\mathbf{S}_i \leq \mathbf{T}_j\}$, this event becomes $\{\mathbf{S}_i \leq t_j\}$ and we obtain $p_{ij} = F((t_j - \mu_i)/ \sigma_i)$, (see Figure 3).

b) Adams and Messick then observed that $z_{ij} = F^{-1}(p_{ij}) = (t_j - \mu_i)/ \sigma_i$ verifies a system of relations of the form

$$z_{kj} = a_{kl}\, z_{lj} + b_{kl} \qquad [C_{AM}]$$

with positive coefficients a_{kl} for any pair of indices of rows l, k =1,..., I and columns j, (by taking $t_0 = -\infty$ and $t_J = \infty$). Regarding the table of data z_{kj} this results in the verifiable fact that we pass from any row k to any other row of index l by a positive linear transformation that depends only on indices k and l.

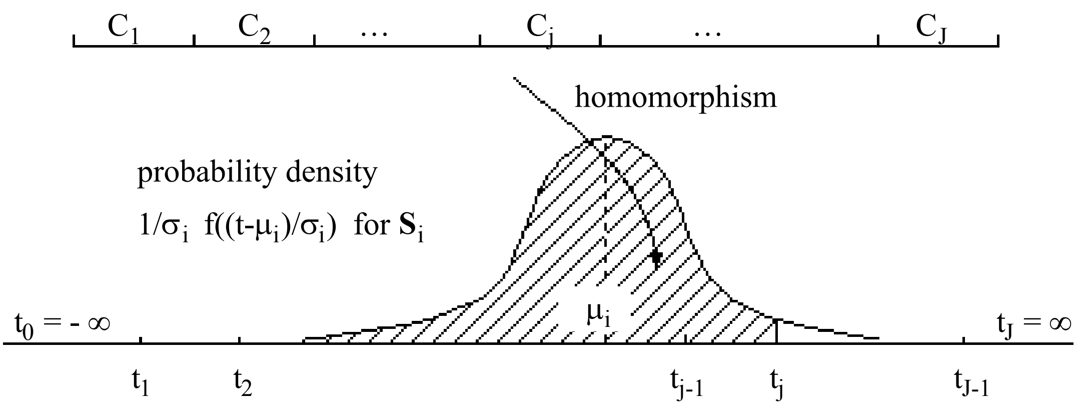

Figure 3, the numerical mode of representation of successive intervals.

c) Relations $[C_{AM}]$ are equivalent to $p_{ij} = F((t_j - \mu_i)/\sigma_i)$ but they are expressed intrinsically with only p_{ij} and z_{ij} without using or mentioning representations μ_i, σ_i and t_j of stimulations s_i and categories C_j. Adams and Messick used these relations as an empirical relational system in the table of p_{ij} (or their transform z_{ij}), and by starting out from $[C_{AM}]$, they established a theorem of representation that demonstrates the existence of $\mu_i = \mu(s_i)$ as a measurement scale of the stimuli, and the existence of t_j as the second scale for the bounds of the intervals. They also established a theorem of characterisation to show that it is these two interval scales that undergo the same transformation of the sub-group of positive linear transformations.

d) The data are the numbers n_{ij} of the contingency table and we calculate the empirical cumulated frequencies $q_{ij} = (\sum_{r=1\ldots j} n_{ir})/n_{i+}$ of the responses in the combination of categories

C_1 U… U C_j for each s_i, (with $n_{i+} = \sum_{r=1\ldots J} n_{ir}$). The q_{ij} are the estimators of p_{ij} and it is with them and the transforms $z^e_{ij} = F^{-1}(q_{ij})$ that we must evaluate relations [C_{AM}] or the simplest equivalent relations $z^e_{ij} = a_i\ t_j + b_i$ with $a_i = 1/\sigma_i$ et $b_i = -\ \mu_i/\sigma_i$. Naturally, following the theoretical considerations on the p_{ij}, it is now necessary to present a function of distribution F for the evaluation.

e) We know that data, and thus here z^e_{ij}, are generally subject to disturbance and that they cannot verify the equivalent relations of [C_{AM}]. Therefore it is necessary to apply a probabilistic measurement technique (Maurin 1986 b). On the theoretical level we have developed a hypothesis test to accept the hypothesis of a measurement scale, and on the practical level, when the hypothesis is accepted, the $z^e_{ij} = a_i\ t_j + b_i$ form a system with 2I + J-1 unknowns a_i, b_i, t_j , and relations I (J-1), to which two additional conditions must be added to set the coefficients of the common interval scale.
This system is overdetermined as soon as (I-1) (J-3) is positive, which is always verified in practice with two stimulations or more and four categories or more, and cannot be used directly for a numerical resolution. Thus we set the term sum of squares errors

$$Q_F = \sum_{ij} \{z^e_{ij} - a_i\ t_j - b_i\}^2$$

which is identically null under conditions [C_{AM}] for z^e_{ij}, and in the presence of noise in the data it is possible, classically, to seek the values of a_i, b_i and t_j for which Q_F is minimal (as has been done already for the law of categorical judgements, § V.2, Torgerson).

f) When implementing successive intervals the choice of law F and the optimisation of Q_F are done together. To do this we simply repeat the minimisation for several laws and use that which leads to the lowest minimum.
In practice, we limit ourselves to four relatively classical laws with the normal law, the logistic law $F_L(x) = 1/(1 + e^{-x})$, and two dissymmetrical laws to cover the largest possible number of situations, the law of extremes of Fréchet, Fisher and Tippett with $F_{FFT}(x) = \exp(-e^{-x})$, and the law obtained by changing the sign of the law of extremes with $F_{-FFT}(x) = 1 - F_{FFT}(-x)$.

V.4 – Numerical assignment and the relational consequences.
The successive intervals are located at a point of convergence between the measurement and the psychophysics of the categorical responses (figure 4), and when the measurement conditions are accepted this technique is constructive and provides numerical values $\mu_i = -b_i/a_i$ et t_j.

Consequently, it also almost immediately resolves the relation concerned at the origin of the psychophysics. Indeed, each physical stimulation is defined by an intensity x_i in its physical unit. Therefore, knowing the numerical measurement μ_i of any s_i the ensemble of pairs $\{\mu_i , x_i\}$ permits explaining a correspondence between magnitude x and numerised response μ. This corresponds exactly with the notion of the “dose-response” or “stimulus impact” curve between the physical magnitude and the human magnitude, as do Fechner’s and Stevens’ laws, and simple linear regressions with the artificial numberings mentioned.

The relation between the stimulation and the numerised response draws advantage here from the properties of the successive intervals; in particular the differences $\Delta\mu_{i,i+1} = \mu_{i+1} - \mu_i$ are measured on a ratio scale, they are comparable with each other and permit observing the relation's monotony (increase or decrease) as well as variations of monotony with ratios of increase $\Delta\mu_{i+1,i+2} / \Delta\mu_{i,i+1}$ in comparison with the unit.

Consequently, successive intervals are an interesting method with a wealth of properties based on a simple and widely used method of collecting responses. In particular it provides new possibilities in environmental research to establish the transformations between the magnitudes of sources of nuisance and the magnitudes of impact on the population (Maurin 2003 a). This simply requires the collection of responses with scales of ordered categories. Furthermore, the calculation algorithm is simple to use.

V.5 – Multidimensional pursuits.

It should be noted that the theorems of Adams and Messick do not make any specific hypothesis on the algebraic structure of stimuli s_i, meaning that the stimulations studied have a multivariate algebraic structure, with for example a multiple index $s_{\{i1,i2,\ldots,ir\}}$. The resulting numerical measurement is itself multivariate $\mu_{\{i1,i2,\ldots,ir\}} = \mu(s_{\{i1,i2,\ldots,ir\}})$ and it is possible to start a simultaneous measurement approach based on successive intervals (Maurin 1986 a, Maurin 2001).

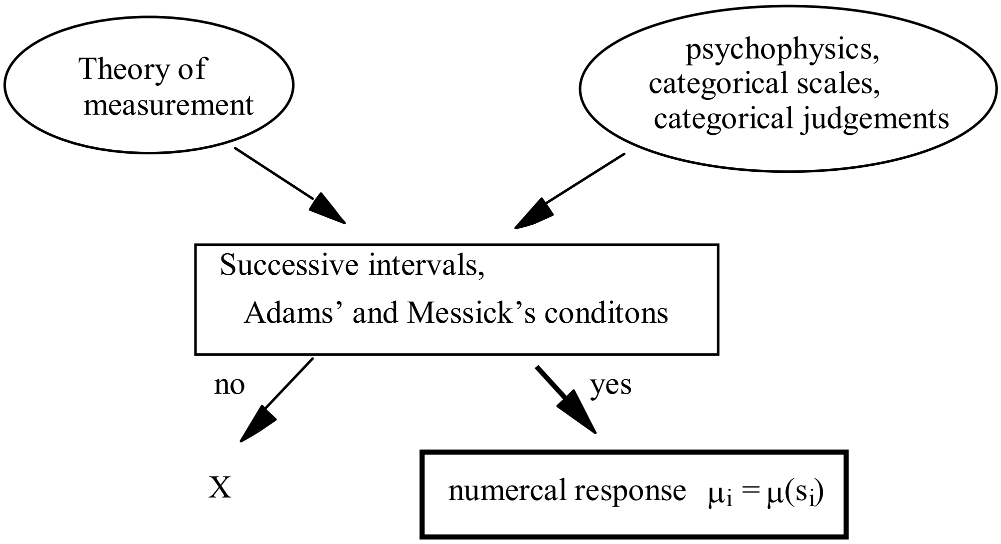

Figure 4, Successive intervals, at the crossroads of psychophysics and measurement.

V.6 - Examples.

Studies on the impact of nuisances and comfort often use responses by categorical scales.

a) During a national survey on nuisances, we collected data on annoyance due to daytime traffic noise on a scale of 4 categories and measured the building frontage noise levels of the persons questioned (a sample of 375 people). The index of the noise used is the equivalent level Leq from 8 a.m. to 8 p.m. which varies from less than 48 dBA to more than 72 dBA (Maurin 2003 b). Since there were few responses in the "annoyed" and "very annoyed" categories for levels lower than 55 dBA, the sections of low levels are grouped together and the final table has seven lines corresponding to exposure values classified by sections of 3 decibels, and four columns for the scale of responses (table 3).

The use of successive intervals gives the numerical values $\mu_L(L_i)$ that are then calibrated by taking $\mu_L(L_1) = 1$ and $\mu_L(L_7) = 7$. The minimum of Q_F is obtained with function F_{-FFT}; the correspondence chart (figure 5) shows a progression of annoyance that accelerates up to 66 decibels A, with a slowing down of increase and a form of saturation beyond this level.

The data fit with the model of successive intervals satisfactorily. However, this is not always the case. For example, the same scale of response in four categories was proposed for annoyance felt at night time at the same time as we measured a nocturnal noise index. However, the measurement axioms are not verified on the resulting contingency table.

	under 55 dBA	56-58	59-61	62-64	65-67	68-70	over 71 dBA
not annoyed	72	46	48	36	24	15	15
slightly annoyed	7	8	8	10	12	10	6
annoyed	1	2	7	9	8	4	7
very annoyed			3	2	6	4	5

table 3, the daytime annoyance-noise contingency table (transposed table); the optimisation algorithm used adapts to the null values.

b) Regarding the comfort of an automotive pedal, we collected a sample of 445 responses on a scale of 4 categories as a function of four factors (each with five modalities): seat height, travel, pedal angle and resistance (Wang & Maurin). Thus we have four contingency tables with five lines and four columns, with for each the possibility of establishing a correspondence between the modalities of a factor and the numerised subjective response.

seat height, mm	C_1	C_2	C_3	C_4
200	9	11	7	3
250	3	24	46	45
300	2	41	59	46
350	5	37	58	19
400	9	11	6	4

Pedal, travel, mm	C_1	C_2	C_3	C_4
100	0	1	8	21
118	1	20	53	44
132	4	45	56	28
152	7	41	51	20
170	16	17	8	4

Tables 4, contingency tables of height-comfort and pedal travel.

The relations [C_{AM}] are accepted while the minima are calculated here using the normal law; two factors show a decreasing curve (pedal travel and resistance) and two show a parabolic shape with a maximum (seat height and pedal angle). It is also possible to calibrate measurements $\mu_h(h_i)$ and $\mu_c(c_i)$ on the same interval scale which gives the same numerical axis for the ordinates (figures 6).

During this study we also sought to compare the results opposite with those obtained when taking raw ratings from 0 to 10 with much more rustic numbering. The comparison shows that the successive intervals lead to finer and more satisfactory interpretations of results.

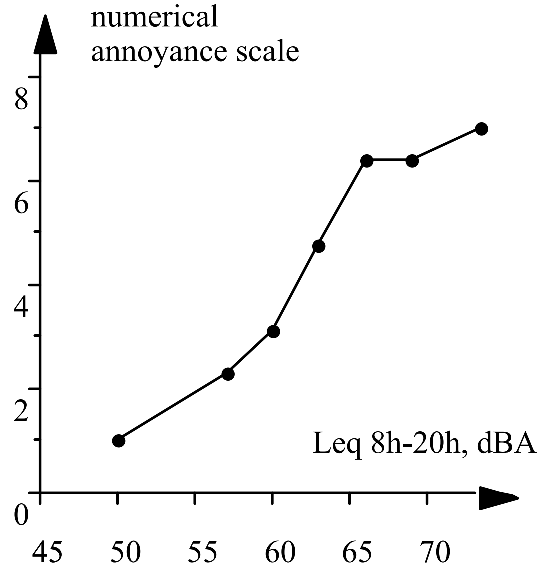

Figure 5 Numerical correspondence between noise levels and annoyance μ

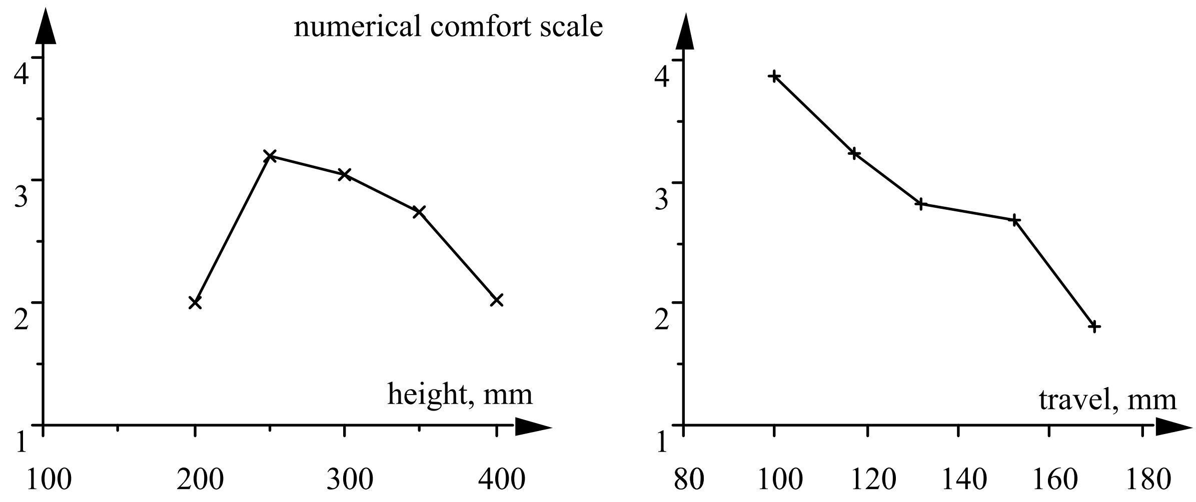

Figure 6 Numerical correspondences between physical factors and comfort μ

VI - Conclusions.

Here, we can quote Largeault "*When describing, one uses a language, a natural language completed or not by mathematical or physical symbols*". It is not our purpose to dictate a language of number in results or plead for such an aim, but it is vital to be aware of the fact that if one uses the number at a given moment, it cannot be done approximately or carelessly with any seriousness. Obviously, care must be taken with the quality of the numbers introduced and the syntax used in the same way as with the grammar of natural languages.
The theory of measurement permits in particular controlling number appeal and the various

temptations it holds, that sometimes beckon the credulous. This theory provides a certain epistemological renovation in how to explain numerical laws correctly. This seems an appropriate place to quote Rabelais' aphorism that "*Science without conscience is but …*" whose sting is known by all (*) and refer to it without succumbing to the spell of these *appeals*.

To end, it should be noted that the extent to which the measurement of successive intervals, among the different methods of collecting responses, endows scales of ordered categories with properties useful in the quest for meaningful numerical and relational methods.

VII – Appendix

As a continuation for a never ended story, we may signal the recent introduction of what is called the "fuzzy logic" implying fuzzy numbers and fuzzy algebra, (1965 for instance). Some years after, other new theories such as evidence theory and possibility theory have been developed, in a sort of generalisation of statistics and probability theory for events. All of these new considerations are intended to allow better approaches to handle both imprecision and uncertainty (Bouchon-Meunier). In a next future, after some time for a maturation period and a golden age for "fuzzy advancements", we may imagine the coming of a fuzzy measurement enlarging the numerical assignment framework, figure 7.

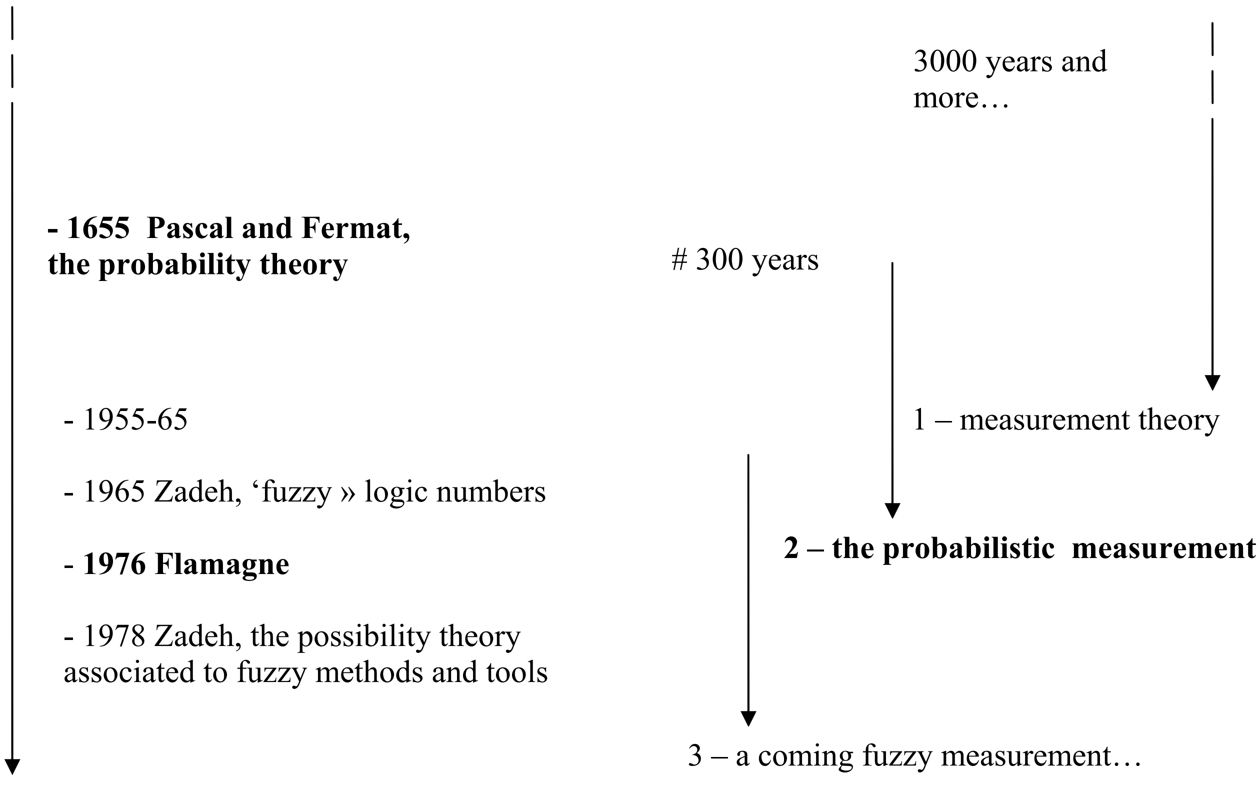

Figure 7, a succession of steps for the measurement approach

* "*…the ruin of the soul.*"

VIII - References.

Adams E., Messick S., 1958, An axiomatic formulation and generalisation of successive intervals scaling, Psychometrika, 23 n°4, 355-368.

Agresti A., 1990, Categorical data Analysis, J. Wiley and sons.

Annett J., 2002, Subjective rating scales; science or art? Ergonomics vol 45 n°14, 966-987

Bouchon-Meunier B., 1995, La logique floue et ses applications, Addison-Wesley France.

Boudon R., 1967, L'analyse mathématique des faits sociaux, Paris, Plon.

Bouyssou D., Pirlot M., 2002, Nontransitive decomposable conjoint measurement, Journal of Math. Psych., vol 46, 677-703.

Coombs C.H., Dawes R.M., Tversky A., 1970, Mathematical psychology, Englewood Cliff, New-Jersey, Prentice-Hall.

de Diéguez M., 1992, Science et Philosophie, Encyclopaedia Universalis vol. 20.

Falmagne J.C., 1976, Random conjoint measurement and loudness summation, Psychological review, 83 n°1, 65-79

Falmagne J.C., 1978 a, A representation theorem for finite random scale systems, J. Math. Psychology, 18 n°1, 52-72.

Falmagne J.C., 1978 b, Probabilistic choice behavior theory: axioms as constraints in optimisation, in Castellane, Restle, Cognitive theory vol 3, L. Elbaum Ass. Pub., 93-114.

Falmagne J.C., 1985, Elements of Psychophysical theory, Oxford Un. Press.

Gescheider G.A., 1997, Psychophysics, the fundamentals, LEA.

Guilbaud G.Th., 1952, Les problèmes de partage, Economie appliquée, vol 5, fasc. 1, 93-137.

Gusdorf G., 1977, De l'histoire des sciences à l'histoire de la pensée, Payot.

Hamerle A., Tutz G., 1980, goodness of fit tests for probabilistic measurement models , J. of Math. Psychology, 21, 153-167.

Kruskal J.B., 1964, Non metric multidimensional scaling : a numerical method, Psychometrika, vol 29, 115-129.

Largeault J., 1992, Description et explication, Encyclopaedia Universalis vol. 17.

Legrand G., 1972, Vocabulaire de la Philosophie, Bordas.

Lévy-Leblond J.M., 1992, Physique et Mathématique, Encyclopaedia Universalis vol. 18.

Luce R.D., Galanter E., 1963, Discrimination, Psychophysical scaling, in Luce, Bush, Galanter, Handbook of mathematical psychology, 193-243, 244-307, J. Wiley and sons.

Luce R.D., Suppes P., 1965, Preference, utility and subjective probability, in Luce, Bush, Galanter, Handbook of math. psychology, vol 3, J. Wiley, 249-410.

Maurin M., 1986 a, Topics on successive intervals, European Mathematical Psychology Group, 17th meeting, Osnabruck.

Maurin M., 1986 b, L'association du Mesurage additif conjoint et des intervalles successifs, Mathématiques et Sciences Humaines, n° 96, 5-29.

Maurin M., 1999, The "logic of levels", What about variances, covariances and linear regression ? Inter-Noise 99, Fort-Lauderdale.

Maurin M., 2001, A non additive conjoint measurement with the successive intervals, the interactions study, European Mathematical Psychology Group, 32th meeting, Lisbonne.

Maurin M., 2003 a, Une analyse structurelle et morphologique des indicateurs d'impact sur l'environnement - Projet PIE, version 2, INRETS-LTE 0301.

Maurin M., 2003 b, Réflexions à propos des relations numériques entre le bruit des transports

et la gêne exprimée, Recherche Transport Sécurité, n° 78, 63-77.
Mausfeld R., 1984, Testing Fechnerian scalibility by maximum likelihood estimation of ordered binomial parameters, in Degreef E., Van Buggenhaut J., Trends in Mathematical Psychology, Elsevier Sc. P. B. V., 255-269.
Miller G.A., 1956, The magical number seven, plus or minus two : some limits on capacity for processing information, Psychological review, 63, 81-97.
Perse J., 1992, Psychométrie, Encyclopaedia Universalis vol 19.
Pfanzagl J., 1973, Theory of measurement, Physica Verlag, Wurzburg, Wien.
Roberts F.J., 1979, Measurement theory, Reading Massachussetts, Add. Wesley P. Co.
Sheppard R.N., 1962, The analysis of proximities : multidimensional scaling with an unknown distance function, Psychometrika, vol 27, n° 2, 3.
Stevens S.S., 1966, A metric for the social concensus, Science, 151, 530-541
Stevens S.S., 1971, Issues in psychophysics, Psychological review, 78 n° 5, 426-450.
Suppes P., Zinnes J.L., 1963, Basic measurement theory, in Luce, Bush, Galanter, Handbook of math. psychology, vol 1, 1-76, J. Wiley and sons.
Torgerson W.S., 1958, Theory and methods of scaling, J. Wiley and sons.
Wang X., Maurin M., 2003, Discomfort/comfort assessment using an ordered category scale, IEA Congress, Seoul.
Wang X., Monnier G., 2001, Discomfort assessment questionnaire and discomfort parameters, Rapport d'avancement du projet européen REALMAN IST-2000-29357.